Inequality involving altitudes and circumradius of Δ
https://www.linkedin.com/groups/8313943/8313943-6387148134857920513
In any triangle ABC prove that

$$\sum \frac{1}{\sqrt{h_a h_b}} \geq \frac{2}{R}$$

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Let F be area of the triangle ABC. Since $h_a h_b = \dfrac{4\Delta^2}{ab}$ then $\sum \dfrac{1}{\sqrt{h_a h_b}} \geq \dfrac{2}{R} \Leftrightarrow$

$\sum \dfrac{\sqrt{ab}}{2F} \geq \dfrac{2}{R} \Leftrightarrow \sum \sqrt{ab} \geq \dfrac{4F}{R}$. Note that latter inequality, up to notation for area, is inequality of the problem:
https://www.linkedin.com/groups/8313943/8313943-6387147667763445763.
Thus, the following at the same time is solution for both problems.
Since $\sqrt{ab} = 2R\sqrt{\sin A \sin B}$ and $F = 2R^2 \sin A \sin B \sin C$ then $\sum \sqrt{ab} \geq \dfrac{4F}{R} \Leftrightarrow$
(**1**) $\sum \sqrt{\sin A \sin B} \geq 4 \sin A \sin B \sin C$.
By AM-GM Inequality $\sum \sqrt{\sin A \sin B} \geq 3\left(\prod \sqrt{\sin A \sin B}\right)^{1/3} = 3(\sin A \sin B \sin C)^{1/3}$.
Thus,remains to prove inequality $3(\sin A \sin B \sin C)^{1/3} \geq 4 \sin A \sin B \sin C \Leftrightarrow$
$(\sin A \sin B \sin C)^{2/3} \leq \dfrac{3}{4} \Leftrightarrow \sin A \sin B \sin C \leq \dfrac{3\sqrt{3}}{8}$.
Applying twice inequality* $\sin x \sin y \leq \sin^2 \dfrac{x+y}{2}, x \in (0,\pi)$ to $(x,y) = (A,B)$ and
$(x,y = (C,\pi/3))$we obtain $\sin A \sin B \leq \sin^2 \dfrac{A+B}{2}$ and $\sin C \sin \dfrac{\pi}{3} \leq \sin^2 \dfrac{C+\pi/3}{2}$.
Then, using this inequality again we obtain that

$$\sin A \sin B \sin C \sin \frac{\pi}{3} \leq \left(\sin \frac{A+B}{2} \sin \frac{C+\pi/3}{2}\right)^2 \leq$$

$$\left(\sin^2 \frac{(A+B)/2 + (C+\pi/3)/2}{2}\right)^2 = \sin^4 \frac{(A+B)/2 + (C+\pi/3)/2}{2} = \sin^4 \frac{\pi}{3}.$$

Hence, $\sin A \sin B \sin C \leq \sin^3 \dfrac{\pi}{3} = \dfrac{3\sqrt{3}}{8}$.
* Proof of inequality $\sin x \sin y \leq \sin^2 \dfrac{x+y}{2}$.

$$2\sin^2 \frac{x+y}{2} - 2\sin x \sin y = 1 - \cos(x+y) - (\cos(x-y) - \cos(x+y)) = 1 - \cos(x+y) \geq 0.$$

Remark.

Although the proof of the inequality $\sin A \sin B \sin C \leq \dfrac{3\sqrt{3}}{8}$ can be reduced to a reference to Jensen's Inequality for function $\ln(\sin x)$ which is concave down on $(0,\pi)$ because $(\ln(\sin x))'' = -\dfrac{1}{\sin^2 x} < 0, x \in (0,\pi)$, nevertheless, I consider the proof given above to be preferable since it does not go beyond elementary trigonometry.